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A DUALITY PRINCIPLE AND CORRESPONDENCE RELATIONS IN ELASTICITY

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Abstract—The equilibrium equations for elasticity problems with field variables depending on two rectangular Cartesian coordinates, x_1 and x_2, reduce to $\sigma_{j1,1}+\sigma_{j2,2}=0, j=1,2,3$. These equations are satisfied if a vector potential $\boldsymbol{\phi}$ is introduced for the stress components σ_{ij}, such that $\sigma_{j1}=\partial\phi_j/\partial x_2$ and $\sigma_{j2}=-\partial\phi_j/\partial x_1$. In terms of the six-dimensional vector field $\boldsymbol{\eta}=[\mathbf{u},\boldsymbol{\phi}]^{\mathrm{T}}$, the basic elasticity equations reduce to

$$\frac{\partial}{\partial x_2}[\mathbf{u},\boldsymbol{\phi}]^{\mathrm{T}}=\mathbf{N}\frac{\partial}{\partial x_1}[\mathbf{u},\boldsymbol{\phi}]^{\mathrm{T}},$$

where $\mathbf{u}$ is the displacement field and $\mathbf{N}$ is called the fundamental elasticity matrix, given by the elastic moduli of the solid. This formulation removes the distinction between the displacement vector field $\mathbf{u}$, and the stress potential vector field $\boldsymbol{\phi}$. Indeed, these two vector fields are, in many respects, each other's dual, in the sense that the solution to a class of "displacement problems" also provides the solution of the corresponding dual "force problems". This duality principle is examined in terms of the known solutions of an edge dislocation and a concentrated line force. Then several correspondence relations are also pointed out and discussed. It is shown that the displacement (stress potential) field of a displacement (stress) boundary-value problem can be reduced to the corresponding stress potential (displacement) field by a simple parameter manipulation.

1. INTRODUCTION

For the sake of specificity, consider plane strain elasticity problems. Then the Airy stress functions of a concentrated force and an edge dislocation can be expressed, respectively, as (Dundurs, 1968)

$$U_{\mathrm{f}}=\frac{P_y}{1+\kappa}[\kappa f(x,y;\delta,\eta)+g(x,y;\delta,\eta)] \tag{1a}$$

$$U_{\mathrm{d}}=-\frac{2\mu b_x}{1+\kappa}[f(x,y;\delta,\eta)-g(x,y;\delta,\eta)], \tag{1b}$$

where δ and η are two constants depending on the elasticity of the considered infinitely extended isotropic solid. Dundurs observes that, when, in eqn (1a), P_y is changed to $2\mu b_x$, and κ to -1, keeping $1+\kappa$ outside the brackets and δ and η inside the brackets unchanged, then eqn (1b) is obtained.

Recently, Ni and Nemat-Nasser (1994) have examined the Dundurs observation in the context of a *general duality principle* which applies to a broad class of elasticity (isotropic or anisotropic) problems, for which the field variables depend only on two rectangular Cartesian coordinate variables, say x_1 and x_2. These authors show that the Dundurs observation is one of several correspondence relations that exist in elasticity. In this brief note, we shall first state the *general duality principle* for a concentrated force and an edge dislocation, and then show several correspondence relations.

2. A DUALITY PRINCIPLE

For plane strain (and anti-plane shear) elasticity problems, the equilibrium equations $\sigma_{i1,1}+\sigma_{i2,2}=0$, $i=1,2,3$, are satisfied if we set $\sigma_{i1}=\partial\phi_i/\partial x_2$ and $\sigma_{i2}=-\partial\phi_i/\partial x_1$, where σ_{ij} is the two-dimensional stress tensor, and ϕ_i is a vector field. It can then be shown that all field equations are satisfied by the solution $\boldsymbol{\eta}=[\mathbf{u},\boldsymbol{\phi}]^{\mathrm{T}}$ of the following system of first-order equations:

$$\frac{\partial\boldsymbol{\eta}}{\partial x_2}=\mathbf{N}\frac{\partial\boldsymbol{\eta}}{\partial x_1}, \tag{2a}$$

where $\mathbf{u}$ is the displacement field and $\mathbf{N}$ is a 6×6 matrix, called the *fundamental elasticity matrix* depending on the elasticity of the medium which may be homogeneous or heterogeneous.† In terms of the elasticity tensor C_{ijkl}, $\mathbf{N}$ is given by

$$\mathbf{N}=\begin{bmatrix}\mathbf{n}_{11} & \mathbf{n}_{12}\\ \mathbf{n}_{21} & \mathbf{n}_{11}^{\mathrm{T}}\end{bmatrix}$$

$$\mathbf{n}_{11}=-[C_{j2k2}]^{-1}[C_{j1k2}]^{\mathrm{T}},\quad \mathbf{n}_{12}=-[C_{j2k2}]^{-1}$$

$$\mathbf{n}_{21}=[C_{j1k1}]-[C_{j1k2}][C_{j2k2}]^{-1}[C_{j1k2}]^{\mathrm{T}}. \tag{2b–e}$$

Let ζ_k, $k=1,2,3$, be the generalized eigenvectors‡ of $\mathbf{N}$, which correspond to the first three eigenvalues with positive imaginary parts, p_k, $k=1,2,3$, and write

$$\zeta_k=\begin{bmatrix}\mathbf{a}_k\\ \mathbf{l}_k\end{bmatrix},\quad k=1,2,3, \tag{3a}$$

where, for each k, a_k and l_k are three-dimensional vectors. Now set

$$\mathbf{A}=[\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3],\quad \mathbf{L}=[\mathbf{l}_1,\mathbf{l}_2,\mathbf{l}_3], \tag{3b,c}$$

and observe that the solution of the problem of an edge dislocation with Burgers' vector $\mathbf{b}$ at the origin is given by (Stroh, 1958; Barnett and Lothe, 1973)

$$\mathbf{u}(x_1,x_2)=\frac{1}{\pi}\operatorname{Im}\left\{\mathbf{A}\left[\sum_{k=1}^{3}\log(x_1+p_kx_2)\mathbf{J}_k\right]\mathbf{L}^{\mathrm{T}}\right\}\mathbf{b} \tag{4a}$$

$$\boldsymbol{\phi}(x_1,x_2)=\frac{1}{\pi}\operatorname{Im}\left\{\mathbf{L}\left[\sum_{k=1}^{3}\log(x_1+p_kx_2)\mathbf{J}_k\right]\mathbf{L}^{\mathrm{T}}\right\}\mathbf{b} \tag{4b}$$

where the principal value of the logarithmic function $\log z$ is defined by $\log z=\log|z|+i\arg(z)$, with $0\leqslant\arg(z)<2\pi$, and $\mathbf{J}_k=[\delta_{ik}\delta_{jk}]$ (k not summed). Moreover, the solution of a line force $\mathbf{f}$ at the origin becomes (Stroh, 1958; Barnett and Lothe, 1973)

†The matrix $\mathbf{N}$ was introduced by Ingebrigtsen and Tonning (1969). Its eigenvalue problem for anisotropic elasticity was examined by Malen (1971) and Barnett and Lothe (1973). The six-dimensional elastic field equations in terms of $\mathbf{N}$ were studied by Chadwick and Smith (1977), and applied to angularly inhomogeneous media by Kirchner (1989) and Ting (1989).

‡For the degenerate cases in which $\mathbf{N}$ admits repeated eigenvalues, the corresponding generalized eigenvectors must be suitably defined, see Chadwick and Smith (1977) or Ni and Nemat-Nasser (1991) for details. For cases where two eigenvalues are nearly equal, see Ting and Hwu (1988).

$$\mathbf{u}(x_1, x_2) = \frac{1}{\pi}\operatorname{Im}\left\{\mathbf{A}\left[\sum_{k=1}^{3} \log(x_1 + p_k x_2)\mathbf{J}_k\right]\mathbf{A}^{\mathrm{T}}\right\}\mathbf{f} \tag{5a}$$

$$\boldsymbol{\phi}(x_1, x_2) = \frac{1}{\pi}\operatorname{Im}\left\{\mathbf{L}\left[\sum_{k=1}^{3} \log(x_1 + p_k x_2)\mathbf{J}_k\right]\mathbf{A}^{\mathrm{T}}\right\}\mathbf{f}. \tag{5b}$$

As is seen from eqns (4) and (5), if we interchange $\mathbf{u}$ and $\boldsymbol{\phi}$, and $\mathbf{b}$ and $\mathbf{f}$, then the solutions are interchanged when $\mathbf{A}$ and $\mathbf{L}$ are also interchanged. Ni and Nemat-Nasser (1994) show that this duality stems from the special symmetric property of the basic six-dimensional equation and the duality of the corresponding auxiliary conditions (boundary conditions), and generally applies to a broad class of problems. For the edge dislocation, the displacement field is discontinuous, whereas the stress field is continuous, as follows:

$$\mathbf{u}(x_1, 0^+) - \mathbf{u}(x_1, 0^-) = -H(x_1)\mathbf{b} \tag{6a}$$

$$\sigma_{i2}(x_1, 0^+) = \sigma_{i2}(x_1, 0^-), \quad i = 1, 2, \tag{6b}$$

where the superscripts + and − denote the values of the corresponding quantity evaluated at the upper and lower faces of the plane $x_2 = 0$, respectively; H is the Heaviside step function. In terms of the vector potential $\boldsymbol{\phi}$, eqn (6b) becomes

$$\boldsymbol{\phi}(x_1, 0^+) = \boldsymbol{\phi}(x_1, 0^{-1}). \tag{6c}$$

In addition, it is required that the strain and rotation vanish at infinity,

$$\frac{\partial \mathbf{u}}{\partial x_i} \to 0, \quad \text{as } |x_j| \to \infty \quad \text{for } i, j = 1, 2, \quad \text{and} \quad x_2 \neq 0. \tag{6d}$$

For a line force at the origin, on the other hand, the displacement field is continuous, while the stress field is unbounded at the origin such that the stress vector potential admits a discontinuity as follows:

$$\mathbf{u}(x_1, 0^+) = \mathbf{u}(x_1, 0^{-1}) \tag{7a}$$

$$\boldsymbol{\phi}(x_1, 0^+) - \boldsymbol{\phi}(x_1, 0^{-1}) = -\mathbf{f}H(x_1). \tag{7b}$$

At infinity it is required that

$$\frac{\partial \boldsymbol{\phi}}{\partial x_i} \to \mathbf{0}, \quad \text{as } |x_j| \to \infty \quad \text{for } i, j = 1, 2, \quad \text{and } x_2 \neq 0. \tag{7c}$$

As is seen, eqns (6) and (7) are dual in the sense that an exchange of $\mathbf{u}$ and $\boldsymbol{\phi}$ reduces one set of the auxiliary conditions to the other set of the auxiliary conditions. Problems for which this property holds are called dual (Ni and Nemat-Nasser 1994).

3. CAVITIES AND RIGID INCLUSIONS

Denote by n_α and t_α, $\alpha = 1,2$, the components of the unit normal and the unit tangent vector of the boundary $\partial\Omega$ of a *cavity* Ω, and choose their directions such that $n_1 = t_2$ and $n_2 = -t_1$. Then the tractions on $\partial\Omega$ become

$$-\mathbf{n}\cdot\boldsymbol{\sigma} = -\frac{\partial \boldsymbol{\phi}}{\partial l} = 0 \quad \text{on } \partial\Omega \text{ of cavity } \Omega, \tag{8}$$

where l measures length along $\partial\Omega$.

Similarly, if Ω is a rigid inclusion, then, in the absence of rigid-body displacement, the gradient of the displacement field $\mathbf{u}$ along $\partial\Omega$ must vanish, i.e.

$$\frac{\partial \mathbf{u}}{\partial l} = \mathbf{0} \quad \text{on } \partial\Omega \text{ of rigid inclusion } \Omega. \tag{9}$$

The boundary conditions (8) and (9) are dual. It can then be concluded from the general duality principle (Ni and Nemat-Nasser, 1994) that the corresponding solutions are also dual; the interchange of $\mathbf{u}$ and $\boldsymbol{\phi}$ and $\mathbf{A}$ and $\mathbf{L}$ interchanges the corresponding solutions. Note that the right-hand side of eqns (8) and (9) may be constant vectors corresponding, respectively, to constant tractions and rigid-body displacements on $\partial\Omega$.

4. CORRESPONDENCE RELATIONS

First consider correspondence relations for the non-degenerate anisotropy. From the solution (4) of an edge dislocation, it is observed that the stress vector potential $\boldsymbol{\phi}$ given by eqn (4b) is obtained from the *corresponding* displacement field in eqn (4a) by just replacing $\mathbf{A}$ by $\mathbf{L}$ in the expression of $\mathbf{u}$. The converse, however, is not valid. Similarly, the displacement field $\mathbf{u}$ of a line force in eqn (5a) results when in the expression of *its own* vector potential $\boldsymbol{\phi}$, i.e. in eqn (5b) $\mathbf{L}$ is replaced by $\mathbf{A}$. Furthermore, the stress vector potential in eqn (4b) of the line dislocation follows from the stress vector potential in eqn (5b) of the line force when, in eqn (5b), $\mathbf{A}$ is replaced by $\mathbf{L}$ and $\mathbf{f}$ is replaced by $\mathbf{b}$. The last correspondence relation is analogous to the Dundurs' results for the isotropic solids.

Now examine the correspondence relations for the isotropic solids. It is known that the solution for a line dislocation $\mathbf{b}$ at the origin in isotropic materials is [e.g. Ni and Nemat-Nasser (1991)]

$$\mathbf{u}(x_1, x_2) = -\frac{1}{\pi}\mathrm{Re}\left\{\mathbf{A}\mathbf{L}^{-1}\boldsymbol{\Lambda}\log(x_1+ix_2)+\frac{\mathbf{A}\mathbf{J}_{12}\mathbf{L}^{-1}\boldsymbol{\Lambda}x_2}{x_1+ix_2}\right\}\mathbf{b} \tag{10a}$$

$$\boldsymbol{\phi}(x_1, x_2) = -\frac{1}{\pi}\mathrm{Re}\left\{\boldsymbol{\Lambda}\log(x_1+ix_2)+\frac{\mathbf{L}\mathbf{J}_{12}\mathbf{L}^{-1}\boldsymbol{\Lambda}x_2}{x_1+ix_2}\right\}\mathbf{b}, \tag{10b}$$

where Λ, for the isotropic case, is a real-valued matrix given by

$$\boldsymbol{\Lambda} = \tfrac{1}{2}[\mathrm{Im}(\mathbf{A}\mathbf{L}^{-1})]^{-1} = 2\mu\begin{bmatrix} \frac{1}{1-\kappa} & 1 & 0 \\ 0 & \frac{1}{1-\kappa} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}, \quad \mathbf{J}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \tag{10c,d}$$

here $\kappa = 3-4\,\nu$ for plane strain.

By the general duality principle, the solution for a line force $\mathbf{f}$ at the origin is obtained from eqn (10a,b):

$$\phi(x_1, x_2) = -\frac{1}{\pi}\mathrm{Re}\left\{\mathbf{L}\mathbf{A}^{-1}\mathbf{Y}\log(x_1+ix_2)+\frac{\mathbf{L}\mathbf{J}_{12}\mathbf{A}^{-1}\mathbf{Y}x_2}{x_1+ix_2}\right\}\mathbf{f} \tag{11a}$$

$$\mathbf{u}(x_1, x_2) = -\frac{1}{\pi}\mathrm{Re}\left\{\mathbf{Y}\log(x_1+ix_2)+\frac{\mathbf{A}\mathbf{J}_{12}\mathbf{A}^{-1}\mathbf{Y}x_2}{x_1+ix_2}\right\}\mathbf{f}, \tag{11b}$$

where $\mathbf{Y}$ is a real-valued matrix, given by

$$\mathbf{Y} = \tfrac{1}{2}[\mathrm{Im}(\mathbf{L}\mathbf{A}^{-1})]^{-1} = -\frac{1}{2\mu}\begin{bmatrix} \frac{\kappa}{1+\kappa} & 0 & 0 \\ 0 & \frac{\kappa}{1+\kappa} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{11c}$$

An examination of the above solutions leads to the following observations:

(i) the displacement field for the line dislocation reduces to its own stress vector potential field if $\mathbf{A}$ is replaced by $\mathbf{L}$, keeping $\mathbf{\Lambda}$ unchanged;
(ii) the displacement field for the line force can be obtained from its own stress vector potential field by simply replacing $\mathbf{L}$ by $\mathbf{A}$, keeping $\mathbf{Y}$ unchanged;
(iii) the stress vector potential field for the line force reduces to the stress vector potential field for the line dislocation if $\mathbf{A}$ is replaced by $\mathbf{L}$ and $\mathbf{Y}$ is replaced by $\mathbf{\Lambda}$;
(iv) the displacement field for the line force is obtained from that of the line dislocation by simply replacing $\mathbf{L}$ by $\mathbf{A}$ and $\mathbf{\Lambda}$ by $\mathbf{Y}$.

For two-dimensional (plane strain) problems, the relevant matrices reduce to

$$\mathbf{A} = \begin{bmatrix} 1 & -i\frac{\kappa}{2} \\ i & -\frac{\kappa}{2} \end{bmatrix}, \quad \mathbf{L} = 2\mu\begin{bmatrix} -i & -\frac{1}{2} \\ 1 & \frac{i}{2} \end{bmatrix}$$

$$\mathbf{\Lambda} = \frac{2\mu}{1+\kappa}\mathbf{I}_2$$

$$\mathbf{Y} = \frac{-\kappa}{2\mu(1+\kappa)}\mathbf{I}_2, \tag{12a–d}$$

where $\mathbf{I}_2$ is the 2×2 identity matrix. In this case, therefore, $\mathbf{\Lambda}$ and $\mathbf{Y}$ are proportional to an identity matrix. Hence, their interchange is affected by simple *scaling* of the corresponding results (keeping κ unchanged here).

It is seen that, if in the expression for $\mathbf{A}$ the parameter κ is replaced† by -1, then it follows that

†This is *not* the same as setting $\kappa = 3-4\nu = -1$ which leads to the unrealistic value of $\nu = 1$ for the Poisson ratio, corresponding to *unstable* materials; the Poisson ratio cannot exceed 1/2.

$$\mathbf{A} \Rightarrow \begin{bmatrix} 1 & \frac{i}{2} \\ i & \frac{1}{2} \end{bmatrix} = \frac{1}{2\mu}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{L}. \tag{13}$$

Hence, this substitution renders $\mathbf{A}$ proportional to the product of a -90 degree rotation matrix and $\mathbf{L}$.

It is seen that if $\mathbf{Y} = -\kappa \mathbf{I}_2/2\mu(1+\kappa)$ is replaced by $\Lambda/4\mu^2 = \mathbf{I}_2/2\mu(1+\kappa)$ (no change in κ here), and that if $\mathbf{A}$ is replaced by the right-hand side of (13) (with $\kappa - 1$ now), then the expression for the stress vector potential of the line force reduces to

$$\phi_f \Rightarrow -\frac{1}{\pi}\mathrm{Re}\left\{\log(x_1+ix_2)+\frac{\mathbf{L}\mathbf{J}_{12}\mathbf{L}^{-1}x_2}{x_1+ix_2}\right\}\frac{1}{(1+\kappa)}\begin{bmatrix} 0 & 1 \\ -1 & 10 \end{bmatrix}\mathbf{f}. \tag{14}$$

This is the stress vector potential of the line dislocation whose Burgers vector is proportional to $(f_2, -f_1)^T$. The relation between the x_2-component of $\mathbf{f}$ and the x_1-component of $\mathbf{b}$ is the Dundurs result. Here, observe that (14) also gives another relation between the x_1-component of the concentrated force and the x_2-component of the Burgers vector, which is not noted in Dundurs (1968).

Note that substitution of $\kappa = -1$ in the expression of $\mathbf{A}$, eqn (13) also produces a correspondence relation between the displacement field and the stress vector potential field of a line dislocation. In view of the aforementioned observation (i), this substitution transforms $\mathbf{u}(x_1, x_2) = (u_1, u_2)^T$ to $(\phi_2, -\phi_1)^T/2\mu$, for the line dislocation. This relation also appears to have been unnoticed before.

The correspondence between the displacement and traction boundary conditions in plane elasticity problems was displayed by Muskhelishvili (1953, pp. 144, 155, eqns 41.1, 41.5), and later pointed out by Sokolnikoff (1956, p. 272, eqn 74.5) who presented a unified formulation of the two sets of boundary conditions. England (1971, pp. 88–90, eqns 4.2, 4.9, 4.10) introduced the factor $-i$ which is equivalent to a $-90°$ rotation and which most closely corresponds to our eqn (14).

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